

# Analytical treatment of SUSY Quasi-normal modes in a non-rotating Schwarzschild black hole.

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## Abstract

We use the Fock-Ivanenko formalism to obtain the Dirac equation which describes the interaction of a massless 1/2-spin neutral fermion with a gravitational field around a Schwarzschild black hole (BH). We obtain approximated analytical solutions for the eigenvalues of the energy (quasi-normal frequencies) and their corresponding eigenstates (quasi-normal states). The interesting result is that all the asymptotic states [and their supersymmetric (SUSY) partners] have a purely imaginary frequency, which can be expressed in terms of the Hawking temperature  $T_H$ :  $E_n^{(\uparrow\downarrow)} = -2\pi i n T_H$ . Furthermore, as one expects for SUSY Hamiltonians, the isolated bottom state has a real null energy eigenvalue.

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## I. INTRODUCTION

The BH can be understood as a thermodynamical system whose (Hawking) temperature and entropy are given in terms of its global characteristics (mass, charge and angular momentum). They are obtained by solving a wave equation for small fluctuations subject to the conditions that the flux be ingoing at the horizon and outgoing at asymptotic infinity. Generally, these conditions lead to a set of discrete complex eigenfrequencies, where the real part represents the frequency of oscillation and the imaginary part represents the damping. The frequencies and damping time of the quasi-normal oscillations called "quasi-normal modes" (QNMs) are determined only by the black hole's parameters and are independent of the initial perturbations. It should be possible to infer the black hole parameters solely from the QNM frequencies. QNMs carry information of black holes and neutron stars, and thus are also of great importance to gravitational-wave astronomy[1]. These oscillations are mainly produced during the formation phase of compact stellar objects and can be strong enough to be detected by several large gravitational wave detectors under construction. Recently, QNMs of particles with different spins in black hole spacetimes have also received much attention[2].

Numerical relativity has provided evidence that the QNMs dominate the gravitational-wave signal associated with many processes involving dynamical black holes (such as the formation of black holes in gravitational collapse or binary merger). Since the QNMs encode information concerning the parameters of the black hole one may hope that the gravitational-wave detectors that are now coming into operation will be able to use these signals to investigate the black-hole population of the universe. Even though most studies of QNMs have been motivated by their potential astrophysical relevance, there are several other reasons why one might be interested in understanding the spectrum of oscillations of a black hole. In particular, the modes have played a role in discussions of black-hole stability[3]. Recent investigations of large extra dimensions, has led to the somewhat striking prediction that mini black holes may be observed at particle accelerators such as the LHC[4]. One problem is that in order to suppress a rapid proton decay we need to physically split the quarks and leptons. Such models are generically called split fermion models[5]. Most methods in evaluating the QNMs are numerical. Recently, the Dirac field QNMs were evaluated for a Schwarzschild black hole[6]. A powerful WKB scheme was devised by Schutz and Will[7],

and it was also extended to higher orders[8]. In particular, the interaction of fermions with a gravitational field is an important topic which was firstly studied by Brill and Wheeler[9] in the 1950s, and later by many other authors[10]. It is well known that the vector field equation in a Schwarzschild spacetime can be separated and the time and angular equation integrated. One is then left with the separated radial equation whose integration is not immediate. The remaining Schrödinger-like equation has received great attention and provides QNMs.

On the other hand, the neutrino does not respond directly to electric or magnetic fields. Therefore, if one wishes to influence its orbit by forces subject to simple analysis, one has to make use of gravitational fields. In other words, one has to consider the physics of a neutrino in a curved metric. In this work we revisit this topic using some ideas of the Fock-Ivanenko formalism and SUSY to study the radiation of energy produced by a neutrino, which is affected by a gravitational field produced by a Schwarzschild BH. To do so it we shall study approximated analytical solutions for the Dirac equation for a massless  $1/2$ -spin-fermion around a neutral and non-rotating Schwarzschild BH. This topic has been recently studied using numerical methods, in a  $d + 1$  spacetime, where the  $d - 2$  extra dimensions are compact[11]. In this paper we expand the formalism there developed, making use of the supersymmetrical (SUSY) properties of the Schrödinger equation.

The paper is organized as follows: in Sect. II we obtain the Dirac equation for a  $1/2$ -spin massless fermion without charge (neutrino) in a static and spherically symmetric spacetime, using the Fock-Ivanenko formalism. In Sect. III we obtain the radial equation for the spinors on such a metric. The spinors obey Schrödinger-like stationary equations. In particular, in Sect. IV we study these equations for a Schwarzschild BH. In Sect. V we study the SUSY properties of the effective radial potentials of the Schrödinger-like equations. In Sect. VI we obtain approximated solutions of the quasi-normal frequencies and their eigenstates and SUSY partners for the spinors. In both asymptotic cases, the neutrinos are close, and very distant to the BH. The quantization of quasi-normal modes is studied in the appendix. In Sect. VII we have included some comments about the treatment of  $d > 4$ - neutrinos where the additional dimensions are compact. Finally, in Sect. VIII we develop the final comments.

## II. DIRAC EQUATION FOR A NEUTRINOS IN A STATIC AND SPHERICALLY SYMMETRIC SPACETIME

We consider the metric that describes a static and spherically symmetric spacetime

$$d\bar{s}^2 = f(r)dt^2 - f^{-1}(r)dr^2 + r^2 d\Omega^2, \quad (1)$$

where  $\Omega$  is the solid angle, such that  $d\Omega^2 = \sin^2(\theta)d\theta^2 + \cos^2(\theta)d\phi^2$ . In this paper we shall use natural units:  $\hbar = c = 1$ . The Dirac equation for a free massless fermion  $\Psi$  with spin  $1/2$ , on the metric (1), is

$$\bar{\gamma}^\mu \bar{\nabla}_\mu \bar{\Psi} = 0, \quad (2)$$

where  $\bar{\nabla}_\mu$  denotes the  $\mu$ th component of the covariant derivative and  $\bar{\gamma}^\mu$  are the Dirac matrix components on the metric (1). Using conformal transformations, one obtains

$$\bar{g}_{\mu\nu} \rightarrow g_{\mu\nu} = \Theta^2 \bar{g}_{\mu\nu}, \quad (3)$$

$$\bar{\Psi} \rightarrow \Psi = \Theta^{-3/2} \bar{\Psi}, \quad (4)$$

$$\bar{\gamma}^\mu \bar{\nabla}_\mu \bar{\Psi} \rightarrow \Theta^{5/2} \gamma^\mu \nabla_\mu \Psi, \quad (5)$$

where we shall take  $\Theta = 1/r$ , so that  $\Psi = r^{3/2} \bar{\Psi}$ . The resulting metric is

$$ds^2 = \frac{f}{r^2} dt^2 - \frac{1}{f r^2} dr^2 - d\Omega^2, \quad (6)$$

on which

$$\gamma_\mu \nabla^\mu \Psi = 0. \quad (7)$$

The spinors  $\chi^{(\uparrow\downarrow)}(\theta, \phi)$  on the 2-sphere obey the equation<sup>1</sup>[12]

$$\gamma_i \nabla^i \chi_l^{(\uparrow\downarrow)} = i \begin{pmatrix} (l+1) \\ (-l) \end{pmatrix} \chi_l^{(\uparrow\downarrow)} = i (\kappa_{\uparrow\downarrow}) \chi_l^{(\uparrow\downarrow)}, \quad (8)$$

where

$$\kappa_{\uparrow\downarrow} = \begin{pmatrix} l+1 \\ -l \end{pmatrix}. \quad (9)$$

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<sup>1</sup> Notation: in what follows we shall use the compact notation denoting by  $\uparrow$  all that correspond with the spinor "up", and with  $\downarrow$  all that correspond with the spinor "down". For instance, the scalar  $\kappa_\uparrow = l+1$  corresponds to  $\chi^{(\uparrow)}$  and  $\kappa_\downarrow = -l$  corresponds to  $\chi^{(\downarrow)}$ .

Notice that  $(\kappa_{\uparrow\downarrow})$  can take integer values, but the zero is excluded:  $(\kappa_{\uparrow\downarrow}) \neq 0$ . If we expand the function  $\Psi$  as

$$\Psi = \sum_l \left( \phi_l^{(\uparrow)} \chi_l^{(\uparrow)} + \phi_l^{(\downarrow)} \chi_l^{(\downarrow)} \right), \quad (10)$$

the equations of motion for  $\phi_l^{(\uparrow)}$  and  $\phi_l^{(\downarrow)}$  as a result are found to be [13]

$$\{\gamma^0 \nabla_0 + \gamma^1 \nabla_1 + i(l+1)\gamma^5\} \phi_l^{(\uparrow)} = 0, \quad l = 0, 1, \dots, \quad (11)$$

$$\{\gamma^0 \nabla_0 + \gamma^1 \nabla_1 - i l \gamma^5\} \phi_l^{(\downarrow)} = 0, \quad l = 1, 2, \dots, \quad (12)$$

where we use the spatiotemporal coordinates  $(t, r, \theta, \phi)$  and  $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$  is the interaction. With the aim to solve the equations (11) and (12), we shall make use of the Fock-Ivanenko formalism [14]. To represent the system we choose the following Dirac matrices:

$$\gamma^t = -i \frac{r}{\sqrt{f(r)}} \sigma^3, \quad \gamma^r = r \sqrt{f(r)} \sigma^2. \quad (13)$$

where the  $\sigma^i$  are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (14)$$

so that the spin connections,  $\Gamma_\mu = \frac{1}{4} \gamma^{\nu\mu} \frac{\partial}{\partial x^\nu} g_{\mu\mu}^2$ , are given by<sup>3</sup>

$$\Gamma_t = \sigma^1 \left( \frac{r^2}{4} \right) \frac{d}{dr} \left( \frac{f(r)}{r^2} \right), \quad \Gamma_r = 0. \quad (15)$$

### III. EQUATIONS OF MOTION FOR $\phi_l^{(\uparrow\downarrow)}$

#### A. Spinor $\phi_l^{(\uparrow)}$

The equation of motion for  $\phi_l^{(\uparrow)}$  (the equation for  $\phi_l^{(\downarrow)}$  can be worked out in the same way) is

$$\sigma_2 r \sqrt{f(r)} \left[ \frac{\partial}{\partial r} + \frac{r}{2\sqrt{f(r)}} \frac{d}{dr} \left( \frac{\sqrt{f(r)}}{r} \right) \right] \phi_l^{(\uparrow)} - i\sigma_1 (l+1) \phi_l^{(\uparrow)} = i\sigma_3 \left( \frac{r}{\sqrt{f(r)}} \right) \frac{\partial \phi_l^{(\uparrow)}}{\partial t}. \quad (16)$$

<sup>2</sup> The reader can see, for example, page 113 - chapter 6 - of the book [14].

<sup>3</sup> Notice that the sum does not run over  $\mu$ , but it does over  $\nu$ .

We propose solutions of the form

$$\phi_l^{(\uparrow)}(r, t) = \left( \frac{\sqrt{f(r)}}{r} \right)^{-1/2} e^{-iE_n^{(\uparrow)}t} \begin{pmatrix} i \Psi_1^{(\uparrow)}(r) \\ \Psi_2^{(\uparrow)}(r) \end{pmatrix}, \quad (17)$$

where  $E_n^{(\uparrow)}$  is some constant of integration. With this transformation, the Dirac equation can be written as two coupled first-order equations:

$$f(r) \frac{d\Psi_1^{(\uparrow)}}{dr} - \frac{\sqrt{f(r)}}{r} (l+1) \Psi_1^{(\uparrow)}(r) = E_n^{(\uparrow)} \Psi_2^{(\uparrow)}(r), \quad (18)$$

$$f(r) \frac{d\Psi_2^{(\uparrow)}}{dr} + \frac{\sqrt{f(r)}}{r} (l+1) \Psi_2^{(\uparrow)}(r) = -E_n^{(\uparrow)} \Psi_1^{(\uparrow)}(r). \quad (19)$$

Now we define the tortoise coordinate  $u$  and the function  $W(r)$ , as

$$f(r) \frac{d}{dr} = \frac{d}{du}, \quad W^{(\uparrow)}(r) = \frac{(l+1)}{r} \sqrt{f(r)}, \quad (20)$$

so that the equations (18) and (19) can be written, respectively, as

$$\left[ \frac{d}{du} - W^{(\uparrow)}(u) \right] \Psi_1^{(\uparrow)}(u) = E^{(\uparrow)} \Psi_2^{(\uparrow)}(u), \quad (21)$$

$$\left[ \frac{d}{du} + W^{(\uparrow)}(u) \right] \Psi_2^{(\uparrow)}(u) = -E^{(\uparrow)} \Psi_1^{(\uparrow)}(u). \quad (22)$$

These equations can be separated as

$$\left( -\frac{d^2}{du^2} + V_1^{(\uparrow)}(u) \right) \Psi_1^{(\uparrow)}(u) = [E_n^{(\uparrow)}]^2 \Psi_1^{(\uparrow)}(u), \quad (23)$$

$$\left( -\frac{d^2}{du^2} + V_2^{(\uparrow)}(u) \right) \Psi_2^{(\uparrow)}(u) = [E_n^{(\uparrow)}]^2 \Psi_2^{(\uparrow)}(u), \quad (24)$$

such that the supersymmetric (SUSY) potentials  $V_1^{(\uparrow)}$  and  $V_2^{(\uparrow)}$  are given by the expression

$$V_{1,2}^{(\uparrow)}(u) = \mp \frac{dW^{(\uparrow)}}{du} + [W^{(\uparrow)}(u)]^2. \quad (25)$$

The effective potentials in (25) are supersymmetric. Hence, the functions  $\Psi_1^{(\uparrow)}$  and  $\Psi_2^{(\uparrow)}$  have the same spectrum for quasi-normal modes.

## B. Spinor $\phi_l^{(\downarrow)}$

In the same manner one can obtain the Schrödinger-like equations once we propose the solutions

$$\phi_l^{(\downarrow)}(r, t) = \left( \frac{\sqrt{f(r)}}{r} \right)^{-1/2} e^{-iE_n^{(\downarrow)}t} \begin{pmatrix} i \Psi_1^{(\downarrow)}(r) \\ \Psi_2^{(\downarrow)}(r) \end{pmatrix}, \quad (26)$$

such that the superpotential  $W^{(\downarrow)}(r)$  is

$$W^{(\downarrow)}(r) = -\frac{l}{r}\sqrt{f(r)}, \quad (27)$$

and the Schrödinger-like equations for  $\Psi_1^{(\downarrow)}$   $\Psi_2^{(\downarrow)}$  are

$$\left(-\frac{d^2}{du^2} + V_1^{(\downarrow)}(u)\right) \Psi_1^{(\downarrow)}(u) = [E_n^{(\downarrow)}]^2 \Psi_1^{(\downarrow)}(u), \quad (28)$$

$$\left(-\frac{d^2}{du^2} + V_2^{(\downarrow)}(u)\right) \Psi_2^{(\downarrow)}(u) = [E_n^{(\downarrow)}]^2 \Psi_2^{(\downarrow)}(u), \quad (29)$$

with the potentials  $V_{1,2}^{(\downarrow)}(u)$  given by

$$V_{1,2}^{(\downarrow)}(u) = \mp \frac{dW^{(\downarrow)}}{du} + [W^{(\downarrow)}(u)]^2. \quad (30)$$

The equations (23-24) and (28-29) give us the radial information about the wave functions of the spinors  $\phi_i^{(\uparrow,\downarrow)}(r, t)$ . In the following sections we shall consider an analytical treatment to solve the pairs of Schrödinger-like equations (23-24) and (28-29), for a Schwarzschild black hole.

#### IV. NEUTRINOS IN A SCHWARZSCHILD BLACK-HOLE

We consider the particular case where the function  $f(r)$  in the metric (1) is given by

$$f(r) = 1 - \frac{a}{r}, \quad (31)$$

such that  $a = 2GM$  is the Schwarzschild radius. In this case the tortoise coordinate  $u(r)$  is given by

$$u(r) = r + a \ln \left[ \frac{r}{a} - 1 \right], \quad \text{or,} \quad u(r) = \frac{a}{1 - f(r)} + a \ln \left[ \frac{f(r)}{1 - f(r)} \right]. \quad (32)$$

From the second equation in (32), we obtain

$$[1 - f(u)] = \frac{1}{1 + LW(e^{(u-a)/a})}, \quad (33)$$

where  $LW(x)$  is the Lambert function, such that  $LW(x) e^{LW(x)} = x$  and  $LW(0) = 0$ . Now we are aimed to obtain the potentials  $V_{1,2}^{(\pm)}(u)$  as a function of  $u$ . From eqs. (20), the second equation of (32) and (33), we obtain the superpotentials

$$W^{(\uparrow\downarrow)}(u) = \frac{(\kappa_{\uparrow\downarrow})}{a} \frac{[LW[e^{(u-a)/a}]]^{1/2}}{[1 + LW[e^{(u-a)/a}]]^{3/2}}. \quad (34)$$

The exact potentials, written as a function of  $u$ , are

$$\begin{aligned} V_{1,2}^{(\uparrow\downarrow)}(u) &= [W^{(\uparrow\downarrow)}]^2(u) \pm \frac{dW^{(\uparrow\downarrow)}(u)}{du} = \frac{(\kappa_{\uparrow\downarrow})^2}{a^2} \frac{LW[e^{(u-a)/a}]}{[1 + LW[e^{(u-a)/a}]]^3} \\ &\mp \frac{1}{2} \frac{(\kappa_{\uparrow\downarrow})}{a^2} \frac{[2LW[e^{(u-a)/a}] - 1]}{[1 + LW[e^{(u-a)/a}]]^{7/2}} \sqrt{LW[e^{(u-a)/a}]}. \end{aligned} \quad (35)$$

In the limit case where the particle is close to the Schwarzschild radius:  $u \rightarrow -\infty$ , so that  $e^{(u-a)/a} \rightarrow 0$ . In this case the effective potentials can be approximated by

$$V_{1,2}^{(\uparrow\downarrow)}(u) \simeq \frac{(\kappa_{\uparrow\downarrow})^2}{a^2} e^{(u-a)/a} \pm \frac{(\kappa_{\uparrow\downarrow})}{2a^2} e^{(u-a)/(2a)}. \quad (36)$$

In order to make a dimensionless description of the problem we can make the following transformation:  $2v = -(u - a)/a$ , and the Schrödinger-like equations (23) and (24), for  $\Psi_1^{(\uparrow\downarrow)}(v)$  and  $\Psi_2^{(\uparrow\downarrow)}(v)$ , close the Schwarzschild radius, become

$$\left\{ -\frac{d^2}{dv^2} + 4(\kappa_{\uparrow\downarrow})^2 e^{-2v} \pm 2(\kappa_{\uparrow\downarrow}) e^{-v} - 4a^2 \left( \frac{[E_n^{(i,\uparrow\downarrow)}]^2}{[E_n^{(j,\uparrow\downarrow)}]^2} \right) \right\} \begin{pmatrix} \Psi_i^{(\uparrow\downarrow)}(\alpha, l, v) \\ \Psi_j^{(\uparrow\downarrow)}(\beta, l, v) \end{pmatrix} = 0, \quad (37)$$

where  $E_n^{(i,\uparrow\downarrow)} = E_n^{(j,\uparrow\downarrow)}$  and

$$i = 1, \quad j = 2 \quad \alpha = n + 1, \quad \beta = n, \quad \text{for} \quad \phi_l^{(\uparrow)}, \quad (38)$$

$$i = 2, \quad j = 1 \quad \alpha = n, \quad \beta = n + 1, \quad \text{for} \quad \phi_l^{(\downarrow)}. \quad (39)$$

Notice that for large  $r$  one obtains  $u \rightarrow -\infty$ , so that  $V_{1,2}^{(\uparrow\downarrow)} \rightarrow 0$ . In other words, the approximated potential (36) becomes null for large distances and takes the same asymptotic value as the exact potential (35). Therefore, the asymptotic solutions of the eq. (37) will be a good approximation for  $r \rightarrow \infty$ . However, the approximated potential (36) is not a good approximation for intermediate distances.

## V. SUPERSYMMETRY OF HAMILTONIANS

We consider the Hamiltonians  $H_1^{(\uparrow\downarrow)}$  and  $H_2^{(\uparrow\downarrow)}$

$$H_1^{(\uparrow\downarrow)} = -\frac{d^2}{dv^2} + [W^{(\uparrow\downarrow)}]^2(v) - \frac{dW^{(\uparrow\downarrow)}(v)}{dv}, \quad (40)$$

$$H_2^{(\uparrow\downarrow)} = -\frac{d^2}{dv^2} + [W^{(\uparrow\downarrow)}]^2(v) + \frac{dW^{(\uparrow\downarrow)}(v)}{dv}, \quad (41)$$



such that  $H_2^{(\uparrow\downarrow)} - H_1^{(\uparrow\downarrow)} = 2\frac{dW^{(\uparrow\downarrow)}(v)}{dv}$ . In our case  $W^{(\uparrow\downarrow)}(v) \simeq -2(\kappa_{\uparrow\downarrow})e^{-v/4}$ , so that we obtain  $H_1^{(\uparrow\downarrow)} - H_2^{(\uparrow\downarrow)} = 2W^{(\uparrow\downarrow)}(v)$ . Following Mielnik[15], we shall define the differential operators:

$$b_{(\uparrow\downarrow)}^* = -\frac{d}{dv} + W^{(\uparrow\downarrow)}(v), \quad (42)$$

$$b_{(\uparrow\downarrow)} = \frac{d}{dv} + W^{(\uparrow\downarrow)}(v), \quad (43)$$

then, we see that Hamiltonians (40) and (41) can be rewritten in terms of  $b_{(\uparrow\downarrow)}$  and  $b_{(\uparrow\downarrow)}^*$ :

$$b_{(\uparrow\downarrow)}^* b_{(\uparrow\downarrow)} = b_{(\uparrow\downarrow)} b_{(\uparrow\downarrow)}^* + [b_{(\uparrow\downarrow)}^*, b_{(\uparrow\downarrow)}] = b_{(\uparrow\downarrow)} b_{(\uparrow\downarrow)}^* + 2W^{(\uparrow\downarrow)}(v) = H_2^{(\uparrow\downarrow)} + 2W^{(\uparrow\downarrow)}(v) = H_1^{(\uparrow\downarrow)}. \quad (44)$$

Furthermore,  $H_i^{(\uparrow\downarrow)} b_{(\uparrow\downarrow)}^* = b_{(\uparrow\downarrow)}^* b_{(\uparrow\downarrow)} b_{(\uparrow\downarrow)}^* = b_{(\uparrow\downarrow)}^* H_j^{(\uparrow\downarrow)}$ , which means that

$$H_i^{(\uparrow\downarrow)} \left( b_{(\uparrow\downarrow)}^* \Psi_j^{(\uparrow\downarrow)} \right) = b_{(\uparrow\downarrow)}^* \left( H_j^{(\uparrow\downarrow)} \Psi_j^{(\uparrow\downarrow)} \right) = b_{(\uparrow\downarrow)}^* \left( [E_n^{(\uparrow\downarrow)}]^2 \Psi_j^{(\uparrow\downarrow)} \right) = [E_n^{(\uparrow\downarrow)}]^2 \left( b_{(\uparrow\downarrow)}^* \Psi_j^{(\uparrow\downarrow)} \right). \quad (45)$$

Therefore, if  $\Psi_j^{(\uparrow\downarrow)}(\alpha, l, v)$  is an eigenvector of  $H_j^{(\uparrow\downarrow)}$  with eigenvalue  $[E_n^{(\uparrow\downarrow)}]^2$ , hence  $\Psi_i^{(\uparrow\downarrow)}(\alpha, l, v) = b_{(\uparrow\downarrow)}^* \Psi_j^{(\uparrow\downarrow)}(\beta, l, v)$  will be an eigenvector of  $H_i^{(\uparrow\downarrow)}$  with eigenvalue  $[E_n^{(\uparrow\downarrow)}]^2$ . In other words the potentials  $V_i^{(\uparrow\downarrow)}$  and  $V_j^{(\uparrow\downarrow)}$  possess the same spectra of quasi-normal frequencies (but with different eigenstates), because they are supersymmetric partners derived from the same superpotentials  $W^{(\uparrow\downarrow)}(v)$ . Notice that in cases where  $E_n^{(\uparrow\downarrow)}$  is not real, the Hamiltonians will not be hermitian.

## VI. APPROXIMATED SOLUTIONS

In order to solve the equations (37) we must consider that such equations are only valid close to the Schwarzschild horizon:  $r \geq a$ . In this limit,  $v \rightarrow \infty$ . We shall consider separately the cases for the potentials  $V_1^{(\uparrow\downarrow)}$  and  $V_2^{(\uparrow\downarrow)}$ , for the spinors  $\phi_l^{(\uparrow\downarrow)}$ .

### A. Spinor $\phi_l^{(\uparrow)}$

As a first case we consider the radial equations for the spinor  $\phi_l^{(\uparrow)}$ . We shall consider, separately, the attractive and repulsive potentials in the equation (25).

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<sup>4</sup> Actually, there exist a family of solutions  $W^{(\uparrow\downarrow)}(v) = \frac{C[2(\kappa_{\uparrow\downarrow})e^{-v} \text{Ei}(1, -4(\kappa_{\uparrow\downarrow})e^{-v}) + 2(\kappa_{\uparrow\downarrow})e^{-v}]}{C \text{Ei}(1, -4(\kappa_{\uparrow\downarrow})e^{-v}) + 1}$ , for  $\text{Ei}(az) = z^{a-1} \Gamma(1-a, z)$ , which give us SUSY potentials  $V_{1,2}^{(\uparrow\downarrow)}(v) = 4(\kappa_{\uparrow\downarrow})^2 e^{-2v} \pm 2(\kappa_{\uparrow\downarrow})e^{-v}$ , but in this paper we shall restrict our study to the particular solution with  $C = 0$ .

1. *Attractive potential for  $\phi_l^{(\dagger)}$*

The first case consists in

$$V_2^{(\dagger)}(l, v) = 4(l+1)^2 e^{-2v} - 2(l+1) e^{-v}, \quad (46)$$

which is similar (but not exactly equal) to a Morse potential<sup>5</sup>. The general solution for (37) is given by

$$\Psi_2^{(\dagger)}(n, l, v) \simeq e^{-v/2} [\mathcal{I}_{\nu_1}[z(v)] - \mathcal{I}_{\nu_2}[z(v)]] \left[ B_{nl} + C_{nl} \int \frac{dv e^v}{[\mathcal{I}_{\nu_1}[z(v)] - \mathcal{I}_{\nu_2}[z(v)]]^2} \right], \quad (47)$$

where the  $\mathcal{I}_{\nu_1}[z(v)]$  and  $\mathcal{I}_{\nu_2}[z(v)]$  are the modified Bessel functions of the first kind, with

$$\nu_{1,2} = 2i a E_n^{(\dagger)} \mp \frac{1}{2}, \quad (48)$$

$$z(v) = 2(l+1) e^{-v}. \quad (49)$$

As can be demonstrated [see appendix (A)] the eigenvalues of energy are<sup>6</sup>:

$$E_n^{(2,\dagger)} = -i \frac{n}{2a}, \quad n = 1, 2, \dots \quad (50)$$

and the eigenvalues for the angular moment are  $l \geq 0$ . In order to avoid a possible divergence in (47), we choose  $C_{nl} = 0$ . A particular manifestation of Hamiltonians which are SUSY, is related to the null eigenvalue of energy:  $E_{n=0}^{(\dagger)} = 0$ . In this case  $n = 0$  and the solution is

$$\Psi_2^{(\dagger)}(0, l, v) \simeq B_{0l} e^{-v/2} [\mathcal{I}_{-1/2}[z(v)] - \mathcal{I}_{1/2}[z(v)]] \propto e^{-2(l+1) e^{-v}}, \quad (51)$$

where  $B_{0l}$  is a constant to be determined by normalization.

The  $\Psi_2^{(\dagger)}(n, l, v)$ -partner states can be obtained when we apply the operator of  $b_{\dagger}^*$  to  $\Psi_2^{(\dagger)}(n, l, v)$

$$\Psi_1^{(\dagger)}(n+1, l, v) = \left\{ -\frac{d}{dv} + 2(l+1) e^{-v} \right\} \Psi_2^{(\dagger)}(n, l, v). \quad (52)$$

The correspondence (52) is a manifestation of the SUSY character of the Hamiltonians  $H_1^{(\dagger)}$  and  $H_2^{(\dagger)}$ .

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<sup>5</sup> It is interesting notice that the original bound state Morse potential has the form  $V_{Morse}(v) = S^2 [e^{-2v} - e^{-v}]$  and corresponds to an Hermitian exactly solvable potential with real eigenvalues of energy. However, as was demonstrated in [19], it is possible to extend the usual theory for exactly solvable (ES)[20] and quasi-exactly solvable[21] (QES) potentials to accommodate QNMs solutions.

<sup>6</sup> In general the energy can take the values  $E_n^{(\dagger)} = \mp i n / (2a)$ , but we choose only the eigenvalues with sign minus because they are those that correspond to decaying modes.

## 2. Repulsive potential for $\phi_l^{(\uparrow)}$

Now we consider the potential

$$V_1^{(\uparrow)}(v) = 4(l+1)^2 e^{-2v} + 2(l+1) e^{-v}. \quad (53)$$

In this case the general solution is given by

$$\begin{aligned} \Psi_1^{(\uparrow)}(n+1, l, v) &\simeq \bar{B}_{nl} \left\{ \left[ (l+1) e^{-v/2} + \left( n + \frac{1}{2} \right) e^{v/2} \right] \mathcal{I}_{\mu_1}[z(v)] + e^{-v/2} (l+1) \mathcal{I}_{\mu_2}[z(v)] \right\} \\ &+ \bar{C}_{nl} \left\{ \left[ \left( (n+1) 4^{n+1} 16^{-(n+1)} - \frac{4^{-(n+1)}}{2} \right) e^{v/2} + e^{-v/2} (l+1) 4^{(n+1)} 16^{-(n+1)} \right] \mathcal{I}_{\mu_1}[z(v)] \right. \\ &+ \left. e^{-v/2} 4^{(n+1)} 16^{-(n+1)} (l+1) \mathcal{I}_{\mu_2}[z(v)] \right\} \\ &\times \frac{1}{4} \int \frac{dv e^v}{\left\{ \left[ l+1 + \left( n + \frac{1}{2} \right) e^v \right] \mathcal{I}_{\mu_1}[z(v)] + (l+1) \mathcal{I}_{\mu_2}[z(v)] \right\}^2}, \end{aligned} \quad (54)$$

where

$$\mu_1 = n + 1/2, \quad \mu_2 = n + 3/2, \quad (55)$$

$$z(v) = 2(l+1) e^{-v}. \quad (56)$$

In order to the SUSY expression for the partners (52) to be fulfilled, we shall require that  $\bar{B}_{nl} = 0$ , so that the resulting nonzero constants  $B_{nl}$  in (47) and  $\bar{C}_{nl}$  in (54) should be determined by normalization. As can be demonstrated [see appendix (A)], the energy eigenvalues are

$$E_n^{(1,\uparrow)} = -i \frac{(n+1) - 1}{2a} = -i \frac{n}{2a}, \quad n = 1, 2, \dots \quad (57)$$

Notice that  $E_n^{(2,\uparrow)} = E_n^{(1,\uparrow)}$  [see eq. (37)].

## B. Spinor $\phi_l^{(\downarrow)}$

For completeness, we consider the radial equations for the spinor  $\phi_l^{(\downarrow)}$ , such that the attractive and repulsive potentials are given by (30).

### 1. Attractive potential for $\phi_l^{(\downarrow)}$

Now we consider the attractive potential related to  $\kappa^{(\downarrow)} = -l$ . In this case the potential is

$$V_1^{(\downarrow)}(v) = 4l^2 e^{-2v} - 2l e^{-v}, \quad (58)$$

so that the general solution of (37) is

$$\Psi_1^{(\downarrow)}(n, l, v) \simeq e^{-v/2} \left\{ \alpha_{nl} \left[ \mathcal{I}_{n-1/2} \left[ -2l e^{-v} \right] + \mathcal{I}_{n+1/2} \left[ -2l e^{-v} \right] \right] \right. \quad (59)$$

$$\left. + \beta_{nl} \left[ \mathcal{K}_{n-1/2} \left[ -2l e^{-v} \right] - \mathcal{K}_{n+1/2} \left[ -2l e^{-v} \right] \right] \right\}, \quad (60)$$

such that the functions  $\mathcal{K}_{n \mp 1/2}$  are the modified Bessel functions of second kind, and the quasi-normal frequencies are

$$E_n^{(1, \downarrow)} = -i \frac{n}{2a}, \quad n = 0, 1, \dots \quad (61)$$

The particular case with  $n = 0$  give us the bottom energy eigenvalue  $E_{n=0}^{(1, \downarrow)} = 0$ , which corresponds to the eigenfunction

$$\Psi_1^{(\downarrow)}(0, l, v) = \alpha_{0l} e^{-2l e^{-v}}, \quad (62)$$

where we have put  $\beta_{nl} = 0$ , for the eigenfunction to be finite along all the domain  $v$ .

## 2. Repulsive potential for $\phi_l^{(\downarrow)}$

Finally, for completeness, we consider the repulsive potential associated to  $\kappa^{(\downarrow)} = -l$ . The potential has the form

$$V_2^{(\downarrow)}(v) = 4l^2 e^{-2v} + 2l e^{-v}, \quad (63)$$

and the general solution in this case is

$$\begin{aligned} \Psi_2^{(\downarrow)}(n+1, l, v) &\simeq e^{-v/2} \left\{ \bar{\alpha}_{nl} \left[ \mathcal{I}_{n-1/2} \left[ 2l e^{-v} \right] + \mathcal{I}_{n+1/2} \left[ 2l e^{-v} \right] \right] \right. \\ &\left. + \bar{\beta}_{nl} \left[ \mathcal{K}_{n-1/2} \left[ 2l e^{-v} \right] - \mathcal{K}_{n+1/2} \left[ 2l e^{-v} \right] \right] \right\}. \end{aligned} \quad (64)$$

Finally, the quasi-normal frequencies are

$$E_n^{(2, \downarrow)} = -i \frac{n}{2a} \quad n = 1, 2, \dots \quad (65)$$

In order to fulfill the SUSY expression for the partners

$$\Psi_2^{(\downarrow)}(1, l, v) = \left\{ -\frac{d}{dv} - 2l e^{-v} \right\} \Psi_1^{(\downarrow)}(0, l, v), \quad (66)$$

we shall require that  $\bar{\alpha}_{0l} = 0$  and more generally, that  $\bar{\alpha}_{nl} = 0$  in (64).

### C. Asymptotic solutions for large distances: $r \rightarrow \infty$

To complete our study we shall study the weak gravitational field case for large distances to the BH. In this case  $r \rightarrow \infty$ , so that  $e^{(u-a)/a} \rightarrow e^{u/a}$ . The superpotential can be approximated by

$$W^{(\uparrow\downarrow)}(u)\big|_{u \rightarrow \infty} \simeq \frac{(\kappa_{\uparrow\downarrow})}{a} \frac{1}{LW[e^{u/a}]}, \quad (67)$$

so that the SUSY potentials will be

$$V_{1,2}^{(\uparrow\downarrow)}(u)\big|_{u \rightarrow \infty} \simeq \frac{(\kappa_{\uparrow\downarrow})}{a^2} \frac{1}{\{LW[e^{u/a}]\}^2} [(\kappa_{\uparrow\downarrow}) \mp 1]. \quad (68)$$

For very large distances these potentials tend to 0:  $V_{1,2}^{(\uparrow\downarrow)}(u)\big|_{u \rightarrow \infty} \rightarrow 0$ , in agreement with the approximated potential (36). Hence, the approximated Schrödinger-like equations for  $u \rightarrow \infty$  can be written as

$$-\frac{d^2}{du^2} \Psi_{1,2}^{(\uparrow\downarrow)}(u) \simeq [E_n^{(\uparrow\downarrow)}]^2 \Psi_{1,2}^{(\uparrow\downarrow)}(u), \quad (69)$$

which has a general solution that can be written as a linear combination of  $e^{\pm i E_n^{(\uparrow\downarrow)} u}$ . After taking into account the normalization conditions, and the signature of  $E_n^{(\uparrow\downarrow)}$ , we obtain that for large distances the outgoing solutions are

$$\Psi_{1,2}^{(\uparrow\downarrow)}(u) \sim e^{-nu/(2a)}. \quad (70)$$

This solution agrees perfectly with the whole obtained in [22].

### D. Quasi-normal frequencies and Hawking temperature

The Hawking temperature for the Schwarzschild BH is

$$T_H = \frac{1}{4\pi a}, \quad (71)$$

so that the quasi-normal frequencies can be written as

$$E_n^{(2,\uparrow\downarrow)} = E_n^{(1,\uparrow\downarrow)} = -2\pi i n T_H. \quad (72)$$

This result is exactly whole obtained recently in [11], but using the WKB method (see also [16]).

## VII. NEUTRINOS IN A $d > 4$ -DIMENSIONAL SCHWARZSCHILD BH

The study of neutrinos which are close to multidimensional Schwarzschild BH is an interesting issue. In this case the extended  $d > 4$ -dimensional Schwarzschild metric is given by[23]

$$d\bar{s}^2 = f_{(d)}(r)dt^2 - f_{(d)}^{-1}(r)dr^2 + r^2 d\Omega_{d-2}^2, \quad (73)$$

where  $d$  is the dimension of the spacetime,  $d\Omega_{d-2}^2$  denotes a metric of a  $(d-2)$ -dimensional sphere and

$$f_{(d)}(r) = 1 - \left(\frac{a}{r}\right)^{(d-3)}. \quad (74)$$

In this case the scalars analogous to (9) are

$$\kappa_{\uparrow\downarrow}^{(d)} = \begin{pmatrix} l + \left(\frac{d-2}{2}\right) \\ -[l + \left(\frac{d-4}{2}\right)] \end{pmatrix}, \quad (75)$$

where  $l \geq -(d-2)/2 + 1$  and  $l \geq -(d-2)/2 + 2$  for  $\kappa_{\uparrow}^{(d)}$  and  $\kappa_{\downarrow}^{(d)}$ , respectively. Note that  $d$  only can take integer-even values in order for  $\kappa_{\uparrow\downarrow}^{(d)}$  to be a nonzero integer. The tortoise coordinate can be written as

$$u(r) = a \left[ \frac{1}{1 - f_{(d)}(r)} \right]^{1/(d-3)} {}_2F_1 \left\{ 1, \frac{1}{3-d}; \frac{d-4}{d-3}; \left[ \frac{1}{1 - f_{(d)}(r)} \right]^{1/(d-3)} - 1 \right\}^{(3-d)}, \quad (76)$$

where  ${}_2F_1[\mu, \nu; \gamma; x] = \sum_{n=0}^{\infty} \frac{\nu_n \mu_n x^n}{\gamma_n n!}$  is the hypergeometric function and we have used the fact that

$$(a/r) = \left( \frac{1}{1 - f_{(d)}(r)} \right)^{1/(d-3)}. \quad (77)$$

In this case one can approximate the superpotential for very large extra dimensions, by

$$W^{(\uparrow\downarrow(d))}(r)|_{d \rightarrow \infty} \simeq \frac{(\kappa_{\uparrow\downarrow}^{(d)})}{a \left[ 1 + \left(\frac{u}{a} - 1\right)^{1/(3-d)} \right]} \left[ 1 - \left[ 1 + \left(\frac{u}{a} - 1\right)^{1/(3-d)} \right]^{(3-d)} \right]^{1/2}, \quad (78)$$

The detailed analysis in the SUSY framework is very complicated and goes beyond the scope of this work, but finally one can find that the eigenvalues of the energies are[22]

$$E_n^{(\uparrow\downarrow(d))} = -i \left( \frac{d-3}{2a} \right) n = -2\pi i T_H n, \quad (79)$$

so that the  $d$ -dimensional Hawking temperature increases linearly with the number of extra dimensions:  $T_H = (d-3)/(4\pi a)$ .

## VIII. FINAL COMMENTS

In this paper we have used the Fock-Ivanenko formalism for the Dirac equation in curved spacetime to write the Dirac equation for a massless  $1/2$ -spin-fermion around a Schwarzschild BH. We have calculated approximated analytical solutions for the spinors. The radial eigenfunctions of the spinors  $\phi_l^{(\uparrow\downarrow)}$  are described by Schrödinger-like equations for the radial eigenfunctions of the spinors, which can be rewritten in terms of the redefined tortoise coordinate  $u(r)$ :  $v = -(u - a)/a$ . We have proven the SUSY character of the potentials  $V_1^{(\uparrow\downarrow)}$  and  $V_2^{(\uparrow\downarrow)}$ , corresponding to the Hamiltonians  $H_1^{(\uparrow\downarrow)}$  and  $H_2^{(\uparrow\downarrow)}$ , which are not hermitian. We obtained approximated analytical solutions for eigenvalues of energy (or quasi-normal frequencies). The interesting result here obtained is that all the asymptotic states (and their SUSY partners) have purely imaginary frequencies as eigenvalues of energy, which can be expressed in terms of the Hawking temperature  $T_H$ :  $E_n^{(\uparrow\downarrow)} = -2\pi i n T_H$ . Furthermore, the isolated bottom state has a real null eigenvalue of energy, in agreement with what one expects for a SUSY Hamiltonian[24]. Therefore, all the asymptotic modes decay, except the whole with zero energy. The treatment with  $d > 4$ - compact extra dimensions deserves a more detailed study and goes beyond the scope of this work. However, some comments were included in Sect. VII.

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## Appendix A: Quantization of quasi-normal frequencies: states

We consider the equation (37) for the massless Dirac modes  $\Psi_{1,2}^{(\uparrow\downarrow)}$ . The case with  $\kappa_{\uparrow} = (\ell + 1)$  accounts for the  $\Psi_{1,2}^{(\uparrow)}$ , (with  $\ell = 0, 1, 2, \dots$ ), while  $\kappa_{\downarrow} = -\ell$  corresponds to the  $\Psi_{1,2}^{(\downarrow)}$  ones (of course with  $\ell = 1, 2, \dots$ ).

Although two elections are possible for the potentials (the second exponential term in the potential can take signs  $+$  or  $-$ ), the Schrödinger equation itself seems to come from a

Morse-like potential problem. In symbols:

$$-\frac{d^2\Psi_1^{(\uparrow\downarrow)}}{dv^2} + \{4(\kappa_{\uparrow\downarrow})^2 e^{-2v} + 2(\kappa_{\uparrow\downarrow})e^{-v} - 4a^2[E_n^{(1,\uparrow\downarrow)}]^2\}\Psi_1^{(\uparrow\downarrow)} = 0, \quad (\text{A1})$$

$$-\frac{d^2\Psi_2^{(\uparrow\downarrow)}}{dv^2} + \{4(\kappa_{\uparrow\downarrow})^2 e^{-2v} - 2(\kappa_{\uparrow\downarrow})e^{-v} - 4a^2[E_n^{(2,\uparrow\downarrow)}]^2\}\Psi_2^{(\uparrow\downarrow)} = 0. \quad (\text{A2})$$

Then, we take the way followed by Morse to solve our problem[17]. First, we make a change of variables:

$$y = e^{-v}, \quad \frac{d}{dv} = -y \frac{d}{dy}, \quad \frac{d^2}{dv^2} = y \frac{d}{dy} + y^2 \frac{d^2}{dy^2}. \quad (\text{A3})$$

The resultant equations are

$$\frac{d^2\Psi_1^{(\uparrow\downarrow)}}{dy^2} + \frac{1}{y} \frac{d\Psi_1^{(\uparrow\downarrow)}}{dy} + \left\{ \frac{[\xi_n^{(1,\uparrow\downarrow)}]^2}{y^2} - \frac{2(\kappa_{\uparrow\downarrow})}{y} - 4(\kappa_{\uparrow\downarrow})^2 \right\} \Psi_1^{(\uparrow\downarrow)} = 0, \quad (\text{A4})$$

$$\frac{d^2\Psi_2^{(\uparrow\downarrow)}}{dy^2} + \frac{1}{y} \frac{d\Psi_2^{(\uparrow\downarrow)}}{dy} + \left\{ \frac{[\xi_n^{(2,\uparrow\downarrow)}]^2}{y^2} + \frac{2(\kappa_{\uparrow\downarrow})}{y} - 4(\kappa_{\uparrow\downarrow})^2 \right\} \Psi_2^{(\uparrow\downarrow)} = 0, \quad (\text{A5})$$

where the parameters  $4a^2[E_n^{(i,\uparrow\downarrow)}]^2$  have been replaced by  $[\xi_n^{(i,\uparrow\downarrow)}]^2$ . Finally, a clever transformation of functions bring us closer to a problem with hypergeometric differential equations. Let the set of functions be  $F^{(i,\uparrow\downarrow)}(y)$  such that

$$\Psi_1^{(\uparrow\downarrow)}(y) = e^{-\alpha y} (2\alpha y)^{\frac{\beta}{2}} F_1^{(\uparrow\downarrow)}(y), \quad (\text{A6})$$

$$\Psi_2^{(\uparrow\downarrow)}(y) = e^{-\alpha y} (2\alpha y)^{\frac{\beta}{2}} F_2^{(\uparrow\downarrow)}(y). \quad (\text{A7})$$

At the same time we define the variable  $z$  as

$$z = 2\alpha y, \quad \frac{d}{dy} = 2\alpha \frac{d}{dz}, \quad \frac{d^2}{dy^2} = 4\alpha^2 \frac{d^2}{dz^2}. \quad (\text{A8})$$

Putting it all together, we arrive at the final expression, which entails particular cases of the general confluent hypergeometric differential equation[18]

$$\begin{aligned} & z \frac{d^2 F_1^{(\uparrow\downarrow)}}{dz^2} + (1 + \beta - z) \frac{dF_1^{(\uparrow\downarrow)}}{dz} \\ & + \left\{ \frac{4[\xi_1^{(\uparrow\downarrow)}]^2 + \beta^2}{4z} - \frac{2(\kappa_{\uparrow\downarrow}) + \alpha(\beta + 1)}{2\alpha} + \frac{\alpha^2 - 4(\kappa_{\uparrow\downarrow})^2}{4\alpha^2} z \right\} F_1^{(\uparrow\downarrow)} = 0, \\ & z \frac{d^2 F_2^{(\uparrow\downarrow)}}{dz^2} + (1 + \beta - z) \frac{dF_2^{(\uparrow\downarrow)}}{dz} \\ & + \left\{ \frac{4[\xi_2^{(\uparrow\downarrow)}]^2 + \beta^2}{4z} + \frac{2(\kappa_{\uparrow\downarrow}) - \alpha(\beta + 1)}{2\alpha} + \frac{\alpha^2 - 4(\kappa_{\uparrow\downarrow})^2}{4\alpha^2} z \right\} F_2^{(\uparrow\downarrow)} = 0. \end{aligned}$$



Under certain conditions (which we shall study later), these equations have the form of a confluent hypergeometric differential equation

$$z \frac{d^2 F_{n\beta}(z)}{dz^2} + (1 + \beta - z) \frac{dF_{n\beta}(z)}{dz} + n F_{n\beta}(z) = 0. \quad (\text{A9})$$

The general solution can be expressed in terms of the Laguerre polynomials  $L(n, \beta, z)$

$$F_{n\beta}(z) = A_{n\beta} L(n, \beta, z) + B_{n\beta} \left[ \frac{\Gamma(\beta)}{\Gamma(-n)} \frac{L(n + \beta, -\beta, z)}{\text{bin}(n, \beta) z^\beta} + \frac{\Gamma(-\beta)}{\Gamma(-n - \beta)} \frac{L(n, \beta, z)}{\text{bin}(n + \beta, n)} \right], \quad (\text{A10})$$

where  $\text{bin}(n, \beta) = n! / [\beta!(n - \beta)!]$ . Whatever the rest of the arguments, if  $n$  becomes positive integer, the solution always will be able to be written as a polynomial form (finite number of terms). This treatment of the solutions is particularly useful for the states; and we have the ground states ( $n = 0$ ) using the fact that  $F_{0\beta} = \text{const.}$  in the attractive potentials.

There are four significant cases of interest, which we shall study separately.

*a.*  $\kappa_\uparrow = \ell + 1$  and  $F_1^{(\uparrow)}$

The first case is

$$4[\xi_n^{(1,\uparrow)}]^2 + \beta^2 = 0, \quad (\text{A11})$$

$$-\frac{2(\kappa_\uparrow) + \alpha(\beta + 1)}{2\alpha} = n, \quad (\text{A12})$$

$$\alpha^2 - 4(\kappa_\uparrow)^2 = 0. \quad (\text{A13})$$

A decaying exponential factor in (A6) forces us to set  $\alpha = 2(\ell + 1)$ ; then we obtain  $\beta = -2(n + 1)$ . Finally  $\xi_n^{(1,\uparrow)} = i\frac{\beta}{2} = -i(n + 1)$ . The minus election in the sign of  $\xi$  is because of the energy

$$E_n^{(1,\uparrow)} = \frac{\xi_n^{(1,\uparrow)}}{2a} = -i \frac{(n + 1)}{2a} \quad (\text{A14})$$

which must have a negative imaginary part to recover the outgoing quasi-normal modes. Notice that  $n$  must be a positive integer, and due to the fact  $\beta = -2(n + 1) < 0$ , we obtain  $n = 0, 1, 2, 3, \dots$

*b.*  $\kappa_{\uparrow} = \ell + 1$  and  $F_2^{(\uparrow)}$

In this case

$$4 [\xi_n^{(2,\uparrow)}]^2 + \beta^2 = 0, \quad (\text{A15})$$

$$\frac{2(\kappa_{\uparrow}) - \alpha(\beta + 1)}{2\alpha} = n', \quad (\text{A16})$$

$$\alpha^2 - 4(\kappa_{\uparrow})^2 = 0. \quad (\text{A17})$$

Using similar arguments, but with (A7), we obtain  $\alpha = 2(\ell + 1)$ ,  $\beta = -2n'$  and  $\xi_n^{(2,\uparrow)} = i\frac{\beta}{2} = -in'$ . Again  $n'$  must be a positive integer.

$$E_{n'}^{(2,\uparrow)} = \frac{\xi_{n'}^{(2,\uparrow)}}{2a} = -i\frac{n'}{2a}. \quad (\text{A18})$$

Using the fact that  $\beta = -2n' < 0$ , we obtain  $n' = 1, 2, 3, \dots$ , so that  $n = n' - 1$ , and the energies (A18) can be written as

$$E_n^{(2,\uparrow)} = E_n^{(1,\uparrow)} = -i\frac{(n+1)}{2a}, \quad (\text{A19})$$

where  $n = 0, 1, \dots$

*c.*  $\kappa_{\downarrow} = -\ell$  and  $F_1^{(\downarrow)}$

Only the main results are of interest for us;  $\alpha = 2\ell$ ,  $\beta = -2m$  and  $\xi_m^{(1,\downarrow)} = i\frac{\beta}{2} = -im$ , with  $m$  a positive integer. The energy in this case is

$$E_m^{(1,\downarrow)} = \frac{\xi_m^{(1,\downarrow)}}{2a} = -i\frac{m}{2a}, \quad (\text{A20})$$

where  $m = 1, 2, \dots$

*d.*  $\kappa_{\downarrow} = -\ell$  and  $F_2^{(\downarrow)}$

In this case  $\alpha = 2\ell$ ,  $\beta = -2(m' + 1)$  and  $\xi_{m'}^{(2,\downarrow)} = i\frac{\beta}{2} = -i(m' + 1)$ , with  $m'$  a positive integer. The energy in this case is

$$E_{m'}^{(2,\downarrow)} = \frac{\xi_{m'}^{(2,\downarrow)}}{2a} = -i\frac{(m' + 1)}{2a}. \quad (\text{A21})$$

A similar analysis to those made in (A 0 a) and (A 0 b), gives us  $m' = m - 1$  and finally we find that the energies (or quasi-normal frequencies) have a unique result (we relabel  $m \rightarrow n$ )

$$E_n^{(1,\downarrow)} = E_n^{(2,\downarrow)} = -i\frac{n}{2a}, \quad (\text{A22})$$

where  $n = 1, 2, \dots$

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